Surface Tangents

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Contents

1 Quick solution

For a triangle, defined by points O, A, B, with texture coordinates $O_{(u,v)}$, $A_{(u,v)}$, $B_{(u,v)}$, the tangent vectors defined by the texture coordinates are as follow:

$$\vec{v} = \frac{a_u\vec{b} - b_u\vec{a}}{a_u b_v - b_u a_v}$$

$$\vec{u} = \frac{a_v\vec{b} - b_v\vec{a}}{b_u a_v - a_u b_v}$$

where $\vec{a} = \vec{OA}$, $\vec{b} = \vec{OB}$, $a_{(u,v)} = A_{(u,v)} - O_{(u,v)}$ and $b_{(u,v)} = B_{(u,v)} - O_{(u,v)}$
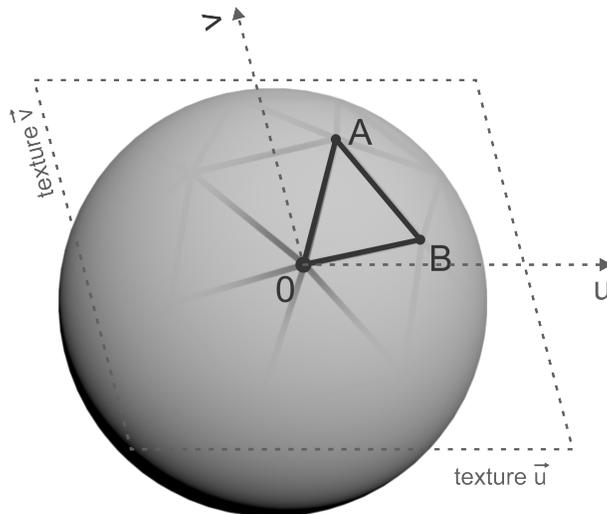

Figure 1: Tangents

2 Introduction

Any point of a surface is most basically defined by its position in 3d space. In addition, a normal vector can provide information about the orientation in which the surface is facing at that point. This is crucial, for example, when calculating lighting.

To expand even more the attributes of a point of a surface, in addition to a position and a normal, two more vectors, tangent to the surface and perpendicular to each other, can define the orientations of the surface's local Cartesian axes. This is used for applying normal mapping on the surface.

3   Let's start

The tangent vectors are two vectors - a tangent and a bitangent, sometimes referred to as tangent and binormal. Generally, they are perpendicular to each other and tangential to the surface, thus also perpendicular to the surface normal. This allows for infinitely many sets of tangent vectors.
However, because they are generally used for normal mapping, we'll look for such tangents set, which is aligned with the axes of the texture mapping's coordinate system.

We have
Polygon vertex points O, A, B
Polygon texture coordinates for each point $O_{(u,v)}$, $A_{(u,v)}$, $B_{(u,v)}$

Let $\vec{a} = \vec{OA} = A - O$
Let $\vec{b} = \vec{OB} = B - O$
Let $a_{(u,v)} = A_{(u,v)} - O_{(u,v)}$
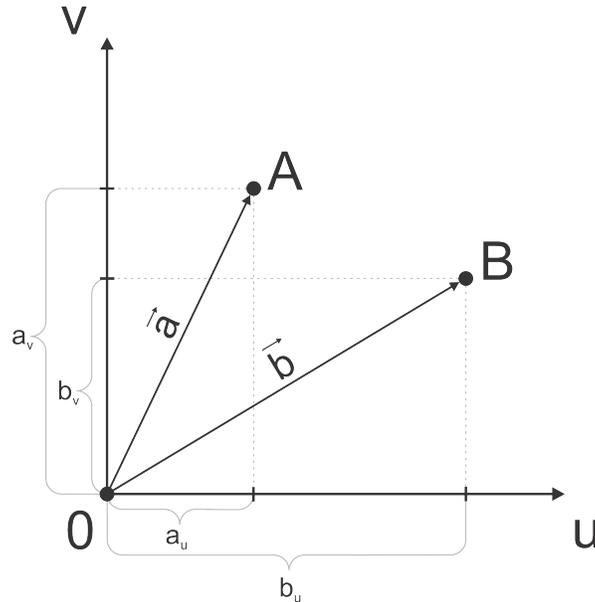Let $b_{(u,v)} = B_{(u,v)} - O_{(u,v)}$

Figure 2: Calculating tangents

4 Calculating tangent vectors

Suppose tangent vectors $\vec{u}$ and $\vec{v}$ such that

- $\vec{u}$ and $\vec{v}$ lie on the plane defined by $\vec{a}$ and $\vec{b}$

- $\vec{u}$ and $\vec{v}$ define a coordinate system, in which $\vec{a}$ is expressed via the coordinates $a_{(u,v)}$ and $\vec{b}$ is expressed via the coordinates $b_{(u,v)}$.

In terms of the coordinate system defined by $\vec{u}$ and $\vec{v}$

$$\vec{a} = a_u \vec{u} + a_v \vec{v} \tag{1}$$

$$\vec{b} = b_u \vec{u} + b_v \vec{v} \tag{2}$$

From (1), express $\vec{u}$ and $\vec{v}$

$$\vec{u} = \frac{\vec{a} - a_v \vec{v}}{a_u} \tag{3}$$

$$\vec{v} = \frac{\vec{a} - a_u \vec{u}}{a_v} \tag{4}$$

From (2), express $\vec{u}$ and $\vec{v}$

$$\vec{u} = \frac{\vec{b} - b_v \vec{v}}{b_u} \tag{5}$$

$$\vec{v} = \frac{\vec{b} - b_u \vec{u}}{b_v} \tag{6}$$

Now equate expressions (3) and (5) to get $\vec{v}$ and expressions (4) and (6) to get $\vec{u}$

$$\frac{\vec{a} - a_u \vec{u}}{a_v} = \frac{\vec{b} - b_u \vec{u}}{b_v}$$

$$\frac{\vec{a} - a_v \vec{v}}{a_u} = \frac{\vec{b} - b_v \vec{v}}{b_u}$$

Solving these yields $\vec{u}$ and $\vec{v}$, expressed in terms of $\vec{a}$, $\vec{b}$ and the texture coordintes, $a_{(u,v)}$ and $b_{(u,v)}$

$$\vec{v} = \frac{a_u \vec{b} - b_u \vec{a}}{a_u b_v - b_u a_v}$$

$$\vec{u} = \frac{a_v \vec{b} - b_v \vec{a}}{b_u a_v - a_u b_v}$$

Note that these vectors may not be perpendicular to each other, as they are determined by how the texture is mapped.

5 Calculating normal vector

The third component, needed to create a typical 3x3 transformation matrix for the tangent space is the normal vector of the surface. The normal vector is perpendicular to the surface and is oriented toward the direction which the surface is facing.

To calculate the normal vector, simply take the cross product of any two non-parallel vectors adjacent to the surface and normalize it.

6 Normal mapping

The two tangent vectors along with the normal vector, define a 3d coordinate system, local for the surface at that point. Having these three vectors as axes of a coordinate system, a typical 3x3 transformation matrix can be constructed, through which any arbitrary point can be transformed into the surface point's local space. This local space is known as tangent space.
When applying normal mapping, typically a normal vector is formed by sampling a texture (the normal map) and converting the red, green and blue color values of range $[0, 1]$ to x, y and z spatial values of range $[-1, 1]$ respectively.

$$mapNormal = mapColor * 2 - 1$$

Thus obtained, the vector is then transformed into the surface's tangent space and the resulting vector acts as a normal for calculating lighting.

$$normal = mapNormal \begin{pmatrix} tangent \\ bitangent \\ normal \end{pmatrix}$$